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Why Total Beta Produces Arbitrary Valuations: A Violation of the "No-Arbitrage" Principle

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Meaningful evaluation equations must be in line with the no-arbitrage principle. If they are not, one can derive any company value from them. This is unacceptable both for practitioners and for academics. This paper shows that total beta does not meet the no-arbitrage principle.

The Problem

Investors that become involved exclusively in privately held companies are ill advised to discount the cash flows of these companies under the guidelines of the capital asset pricing model (CAPM). If one follows this model, all investors are perfectly diversified and only have to carry the systematic risk, but this is precisely untypical of owners of private companies.

It has been repeatedly proposed (not only in this journal) to determine the capital cost of a nondiversified investor with the help of total beta:

$$E[\tilde{r}_j] = r_f + (E[\tilde{r}_m] - r_f)\,\frac{\sigma[\tilde{r}_j]}{\sigma[\tilde{r}_m]}. \qquad (1)$$

The symbols in this equation mean the following: $E[\tilde{r}_j]$ is the expected return of an undiversified investor who invests all her assets in a private company j; r_f is the risk-free rate; $E[\tilde{r}_m]$ represents the expected return on a well-diversified portfolio; $\sigma[\tilde{r}_j]$ denotes the standard deviation of the return of the privately held company; and $\sigma[\tilde{r}_m]$ is the standard deviation of the return on the market portfolio.

There is much debate about the usefulness of Equation (1) for the stated purpose, with the main protagonists including Butler[1] (as a strict proponent) and Kasper[2] and von Helfenstein[3] (as committed opponents).[4] All authors mentioned here are practitioners.[5]

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[1]Peter J. Butler and Keith A. Pinkerton, "Company-Specific Risk—A Different Paradigm: A New Benchmark," *Business Valuation Review* 25 (2006):22–28. Peter J. Butler and Keith A. Pinkerton, "The Butler Pinkerton Model: Empirical Support for Company-Specific Risk," *The Value Examiner* (May/June 2008):32–39. Peter J. Butler and Keith A. Pinkerton, "There Is a 'New Beta' in Town and It's Not Called Total Beta for Nothing!" *Business Valuation Update* (May/June 2008):1–5. Peter J. Butler, "A Response to Unwarranted Fear of Total Beta and the Butler Pinkerton Calculator," *The Value Examiner* (January/February 2010):20–25. Peter J. Butler, "Be Skeptical of Mr. Kasper's Criticisms of Total Beta," *The Value Examiner* (July/August 2010):30–31. Peter J. Butler and Gary S. Schurman, "A Tale of Two Betas," *The Value Examiner* (January/February 2011):21–26. Peter J. Butler, Gary S. Schurman, and Andrew M. Malec, "Practical Evidence and Theoretical Support for Total Beta," *The Value Examiner* (July/August 2011):33–35. Peter J. Butler and Bob Dohmeyer, "The Total Beta Debate: A Real-World Analysis," *Business Valuation Review* 32 (2013):227–230.

[2]Larry J. Kasper, "The Butler Pinkerton Model for Company-Specific Risk Premium: A Critique," *Business Valuation Review* 27 (2008):233–243. Larry J. Kasper, "Total Beta: The Missing Piece of the Cost of Capital Puzzle: A Reply," *Valuation Strategies* (November/December 2009):12–19. Larry J. Kasper, "Fallacies of the Butler-Pinkerton Model and the Diversification Argument," *The Value Examiner* (January/February 2010):8–20. Larry J. Kasper, "A Reply to Messrs. Butler and Pinkerton's Defense of Total Beta and the Butler Pinkerton Calculator," *The Value Examiner* (July/August 2010):23–29. Larry J. Kasper, "Portfolio Theory and Total Beta: A Fairy Tale of Two Betas," *The Value Examiner* (January/February 2012):26–37. Larry J. Kasper, "Total Beta: A Capital Market Analysis with Empirical Evidence," *Business Valuation Review* 32 (2013):212–226.

[3]Sarah B. von Helfenstein: "Revisiting Total Beta," *Business Valuation Review* 28 (2009):201–223. Sarah B. von Helfenstein, "Letters to the Editor: Total Beta Unresolved," *The Value Examiner* (November/December 2011):8–10. Sarah B. von Helfenstein: "Resolving Total Beta," *The Value Examiner* (July/August 2011):23–32. Sarah B. von Helfenstein, "Revisiting Total Beta: Round Two," SSRN eLibrary, accessed at http://dx.doi.org/10.2139/ssrn.1851944, November 4, 2014.

[4]A good introduction into the discussion can be found in Shannon P. Pratt and Roger J. Grabowski, *Cost of Capital: Applications and Examples*, 4th ed. (New York: Wiley & Sons, 2010), Chapter 15.

[5]Dr. Aswath Damodaran is the only prominent academic supporting total beta. See Aswath Damodaran, *Investment Valuation: Tools and Techniques for Determining the Value of Any Asset*, 3rd ed. (New York: Wiley & Sons, 2012), 672–673. This does not prevent him from expressing wrong statements. For example, he asserts that the higher the correlation between returns of the asset and the market (rho), the higher the beta. This is not correct. To best of our knowledge, this has been observed first by Cheremushkin (Sergei Vasilievich Cheremushkin, "How to Avoid Mistakes in Valuation: Guidelines for Practitioners," Working paper, November 26, 2010, accessed at http://dx.doi.org/10.2139/ssrn.1785050, November 4, 2014: "As a result, the assertion made by Professor Damodaran [quoting *Investment Valuation: Tools and Techniques for Determining the Value of Any Asset*, 2nd ed., page 668] that 'the total beta will be higher than the market beta, and will depend upon the correlation between the firm and the market—the lower the correlation, the higher the total beta' is wrong."

Here, we present an argument against total beta that so far has been overlooked. We will show that Equation (1) implies a logical contradiction. More precisely, we demonstrate that the total beta equation cannot hold in an arbitrage-free world. For an academic, this finding is a clear knockout. For a practitioner, the same should be true.

A practitioner may argue that the real world is not free of arbitrage. From time to time, by simply trading assets, currencies, or raw materials, one can make gains even from small differences in prices or interest or currency rates. However, this cannot and will not hold forever, and there are good reasons why no serious economist was ever attempted to state that a market with arbitrages would be an interesting research subject and should be taken seriously. Why? If arbitrage opportunities exist, it can be shown that business values become completely haphazard. There will be not only one, but an infinite number of "reasonable and justified," and hence gratuitous, values of the company. Arbitrary company values are like values plucked out of thin air. Any responsible appraiser will keep away from this.

We are absolutely aware that practical solutions with pure theoretical arguments can only be criticized halfheartedly. If we consider a situation that can theoretically lead to an arbitrage, we have to respect that transaction costs as well as taxes may decrease or even destroy the financial advantage. Is our theoretial argument void by such an observation? We definitely do not think so. Transaction costs, taxes, and other frictions use to be small as compared to the magnitude an arbitrage opportunity can achieve. Therefore, in our opinion, a model with an arbitrage opportunity must also be discarded if one observes significant market frictions.

Total Beta Pricing Equation

In order to prove our assertion, we start with Equation (1), add one on both sides and multiply the result by the price of the privately held company $p[\tilde{CF}_j]$. This yields

$$p[\tilde{CF}_j]\,(1+E[\tilde{r}_j]) = p[\tilde{CF}_j]\left(1+r_f+(E[\tilde{r}_m]-r_f)\,\frac{\sigma[\tilde{r}_j]}{\sigma[\tilde{r}_m]}\right).$$

Using $\tilde{r}_j = (\tilde{CF}_j/p[\tilde{CF}_j] - 1)$ and $p(\tilde{CF}_j)\sigma[\tilde{r}_j] = \sigma[\tilde{CF}_j]$, we get

$$p[\tilde{CF}_j] = \frac{E[\tilde{CF}_j]-(E[\tilde{r}_m]-r_f)\,\dfrac{\sigma[\tilde{CF}_j]}{\sigma[\tilde{r}_m]}}{1+r_f} \qquad (2)$$

by a rearrangement of terms. Let us call this relationship the *total beta pricing equation*. Those who accept total beta must accept this pricing Equation (2), too. We wish to clearly point out that Equation (2) applies irrespectively of how Equation (1) has been derived or of whether it can be justified at all. In this respect, we do not engage in the debate on whether total beta is well founded or not. We simply accept Equation (1) as a dictum, make an easy-to-understand transformation, and arrive at a depiction that is most revealing. This is an important departure from previous discussions on the issue.

The total beta pricing Equation (2) is interesting because it clearly shows the following: In the context of total beta, it is apparently assumed that projects with identical expected cash flows $E[\tilde{CF}_j]$ and identical cash flow risk $\sigma[\tilde{CF}_j]$ must have identical prices $p[\tilde{CF}_j]$. We use this result to demonstrate the logical contradiction. To this end, we give just one example. Any appraisal of a company must take into account that future cash flows are uncertain. This is the case in our example.

Discussion

Let us consider an example to illustrate our point. Suppose we have two companies that are identical in every way except they either operate in the same area but produce different products, or offer identical services but operate in different areas. One could also think of a company that is split into two subcompanies for reasons that have nothing to do with total beta. The first company is denoted by $j = 1$, the second one by $j = 2$. For simplicity, we assume that both companies have identical cash flows:

$$E[\tilde{CF}_1] = E[\tilde{CF}_2] = 1.$$

Let the standard deviations of both companies be

$$\sigma[\tilde{CF}_1] = \sigma[\tilde{CF}_2] = 20\%$$

as well.

We now assume that the cash flows of both companies are correlated and in particular that the correlation coefficient between both cash flows is 70%.[6] Also, let the expected return of the market portfolio be 5% and the riskless rate be 1%. The standard deviation of the market portfolio is assumed to be 20% in order to minimize the effort required to reenact our example. All numbers can be changed without destroying our fundamental argument.

By applying total beta twice, we can evaluate the two firms using the pricing Equation (2). We obtain

[6] From this, we can infer the covariance of the cash flows as $\mathrm{Corr}[\tilde{CF}_1, \tilde{CF}_2] \times \sigma[\tilde{CF}_1] \times \sigma[\tilde{CF}_2] = 0.028$.

$$p[\tilde{C}F_1] = p[\tilde{C}F_2] = \frac{1-(5\%-1\%)\cdot\frac{20\%}{20\%}}{1+1\%} = 0.950495. \quad (3)$$

It is easy to recognize why this is a problem. It is most reasonable to assume that there are no arbitrage opportunities in the market. This means that no one can generate risk-free profits by assembling or disassembling their asset portfolios. Based on this argument, we now look at a new firm that consists of both companies that were previously considered separately. This firm's new cash flow $\tilde{C}F_1 + \tilde{C}F_2$ must have a value of

$$p[\tilde{C}F_1 + \tilde{C}F_2] = p[\tilde{C}F_1] + p[\tilde{C}F_2] = 1.900990.$$

However, using total beta for the new firm delivers quite a different result. Employing Equation (2) yields[7]

$$p[\tilde{C}F_1 + \tilde{C}F_2] = \frac{2-(5\%-1\%)\,\frac{0.36878}{0.2}}{1+1\%} = 1.907172.$$

Admittedly, the difference is small, yet this is simply due to our choice of numbers. We can readily generate more impressive differences. No real company generates a cash flow of just \$1, for instance.

So far, we have only shown that total beta and the no-arbitrage principle are not mutually incompatible. Our previous claims went beyond this: We asserted that it is possible to generate arbitrary company values on the basis of total beta. How so? Just split one subcompany into two sub-subcompanies, and let their cash flows be correlated by less than +1. Choosing appropriate numbers, it is possible to generate values for the new firm that lie either inside or outside the interval [1.900990, 1.907172], as desired. So we obtain not only one, but any value for our privately held firm.

One could argue that in the case of privately held firms, we should not cling to the principle of an arbitrage-free capital market. And, of course, we could and would not force a practitioner to do so. However, when abandoning the principle, we must say goodbye to the CAPM, to option pricing theory, and even to an elementary equation like

$$p[CF_1] = \frac{CF_1}{1+r_f}.$$

All valuation theory rests on the no-arbitrage principle. Practitioners can do as they wish and simply forget about academic principles. Yet, if they use this freedom, they should honestly declare that valuation of companies is just an art that may produce arbitrary, artificial results.

Readers may rest assured that it is easy to construct an unlimited number of examples that show just the same. However, we do not wish to bore our audience.

Academic considerations are often easier to understand if a memorable example is provided. So let us consider a situation in which there are only two possible states one period from now. The chances of either are fifty/fifty. Now, let us look at two different assets. The first one pays \$2 if state number 1 occurs or \$4 if state number 2 occurs. With the second asset, it is just the other way round. The expected cash flows of both assets are the same and amount to $E[\tilde{C}F_1] = E[\tilde{C}F_2] = ½ \times 2 + ½ \times 4 = 3$. Both assets have also identical standard deviations of cash flows, namely, $\sigma[\tilde{C}F_1] = \sigma[\tilde{C}F_2] = \sqrt{(1/2)^2(2-3)^2 + (1/2)^2(4-3)^2} = 1\cdot$

At first glance, paying the same price for these assets seems quite reasonable. If one thinks of coin tossing, with heads or tails as the two possible states, there is apparently no reason why one should come to a different conclusion. Equation (2) tells us exactly this.

Coin tossing aside, let us switch to real economic situations. With state number 1, there will be a boom, while with state number 2, there will be a depression. Market participants may have utility functions that evaluate a dollar during a depression higher than a dollar during a boom. Let the pricing function of all these participants read $p[\tilde{C}F] = ⅓CF(1) + ⅔CF(2)$, where $CF(1)$ and $CF(2)$ denote the cash flows in state number 1 and number 2, respectively.[8] Under this condition, the first asset sells for $p[\tilde{C}F_1] = ⅓ \times 2 + ⅔ \times 4 = 10/3$, while the second sells for $p[\tilde{C}F_2] = ⅓ \times 4 + ⅔ \times 2 = 8/3$. As long as there is someone in the market who buys or sells these assets for the same price by employing Equation (2), there is an arbitrage opportunity. That's the end of the story, which everybody can easily remember.

Note that such a problem cannot happen with the classical CAPM. Although one actually should generally prove such a claim, we are content to demonstrate the truth of our assertion with the numerical example that we have already used above. In order to employ the CAPM equation, we need more information at hand than before, since the CAPM,

[7]The risk of the new firm is evaluated using the covariance. We get $\sigma[\tilde{C}F_1 + \tilde{C}F_2] = \sqrt{(20\%)^2 + (20\%)^2 + 2\cdot 0.028} = 0.36878$.

[8]This, indeed, is a "reasonable" linear pricing function. We call such a function reasonable if it is not possible to achieve risk-free profits by assembling or disassembling one's asset portfolios. In the theory of finance, this result is best known as the fundamental theorem of asset pricing and can nowadays be found in virtually every finance textbook.

$$E[\tilde{r}_j] = r_f + (E[\tilde{r}_m] - r_f)\underbrace{\frac{Cov[\tilde{r}_j, r_m]}{Var[\tilde{r}_m]}}_{=\beta_j}, \qquad (4)$$

requires knowledge of the so-called beta factor or the covariance between the asset to be valued (r_j) and the market portfolio (r_m). However, rearranging the terms yields a pricing equation

$$p[\tilde{CF}_j] = \frac{E[\tilde{CF}_j] - (E[\tilde{r}_m] - r_f)\dfrac{Cov[\tilde{CF}_j, \tilde{r}_m]}{\sigma[\tilde{r}_m]}}{1 + r_f}, \qquad (5)$$

which in its structure is analogous to Equation (2). In order to illustrate our point, we choose a covariance $Cov[\tilde{CF}_j, \tilde{r}_m]$ that leads to the same price that resulted from the total beta approach:

$$0.950495 = \frac{1 - (5\% - 1\%)\cdot\dfrac{Cov[\tilde{CF}_j, \tilde{r}_m]}{20\%}}{1 + 1\%} \quad \Rightarrow$$

$$Cov[\tilde{CF}_j, \tilde{r}_m] = 0.20.$$

This covariance must be the same for both companies because their prices coincide. However, for the combined company, we necessarily have $Cov[\tilde{CF}_1 + \tilde{CF}_2, \tilde{r}_m] = 0.4$ and hence

$$p[\tilde{CF}_1 + \tilde{CF}_2] = \frac{2 - (5\% - 1\%)\cdot\dfrac{0.4}{0.2}}{1 + 1\%} = 1.90099.$$

Using the CAPM for the new firm, the arbitrage opportunity simply vanishes.

In order to do away with any misunderstanding, we want to emphasize that we by no means advocate the use of the CAPM for valuing privately held firms. The CAPM can only be applied if investors are "reasonably well diversified." This is not the case with privately held firms, at least if those companies are small. What is necessary for valuation is rather a modified CAPM where investors deviate from their optimal portfolios and instead are "poorly diversified." How such a model can be developed is (yet) unknown to us. However, in order to gain acceptance, it must have the property (as the standard CAPM) that it will not violate the no-arbitrage principle. Otherwise, it will certainly be dismissed by academia. There is some research that can be pursued; in particular, we have papers in mind that deal with nonmarketability of assets and try to evaluate how the standard CAPM will change under those conditions.[9] Amazingly, the proponents of total beta have ignored this strand of literature so far.

Conclusion

Pricing equations that are not in line with the no-arbitrage principle are useless, since they may produce haphazard company values. We have proven that total beta suffers from such a deficiency. Hence, it should be rejected completely.

Our remarks are solely of a critical nature and do not contain any recommendation about the way in which valuation of privately held firms can be done in a theoretically satisfying manner. However, we feel confident that total beta should be abandoned if privately held firms are to be valued. We also feel sure that the standard CAPM is inappropriate as a solution. To readers who need answers, we come empty-handed, but it could well be that academic papers from the 1970s point in the right direction and can be utilized for a compelling solution.

Ending our contribution with a statement of Leonardo da Vinci seems appropriate: "He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast."

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[9]See in particular David Mayers, "Nonmarketable Assets and Capital Market Equilibrium Under Uncertainty," In Michael C. Jensen, ed., *Studies in the Theory of Capital Markets* (New York: Praeger, 1972), 223–248; David Mayers, "Non-Marketable Assets and the Determination of Capital Asset Prices in the Absence of a Riskless Asset," *Journal of Business* 46 (1973):258–267; David Mayers, "Nonmarketable Assets, Market Segmentation, and the Level of Asset Prices," *Journal of Financial and Quantitative Analysis* 11 (1976):1–12.

Appendix

Derivation from Equation 1 to Equation 2:

$$E[\tilde{r}_j] = r_f + (E[\tilde{r}_m] - r_f)\frac{\sigma[\tilde{r}_j]}{\sigma[\tilde{r}_m]} \tag{1}$$

$$1 + E[\tilde{r}_j] = 1 + r_f + (E[\tilde{r}_m] - r_f)\frac{\sigma[\tilde{r}_j]}{\sigma[\tilde{r}_m]}$$

$$p(\tilde{C}F_j)(1 + E[\tilde{r}_j]) = p(\tilde{C}F_j)\left(1 + r_f + (E[\tilde{r}_m] - r_f)\frac{\sigma[\tilde{r}_j]}{\sigma[\tilde{r}_m]}\right)$$

$$E[\tilde{C}F_j] = p(\tilde{C}F_j)(1 + r_f) + (E[\tilde{r}_m] - r_f)\frac{p(\tilde{C}F_j)\sigma[\tilde{r}_j]}{\sigma[\tilde{r}_m]}$$

$$= p(\tilde{C}F_j)(1 + r_f) + (E[\tilde{r}_m] - r_f)\frac{p(\tilde{C}F_j)\sigma[1 + \tilde{r}_j]}{\sigma[\tilde{r}_m]}$$

$$= p(\tilde{C}F_j)(1 + r_f) + (E[\tilde{r}_m] - r_f)\frac{\sigma\left[p(\tilde{C}F_j)(1 + \tilde{r}_j)\right]}{\sigma[\tilde{r}_m]}$$

$$= p(\tilde{C}F_j)(1 + r_f) + (E[\tilde{r}_m] - r_f)\frac{\sigma[\tilde{C}F_j]}{\sigma[\tilde{r}_m]}$$

$$E[\tilde{C}F_j] - (E[\tilde{r}_m] - r_f)\frac{\sigma[\tilde{C}F_j]}{\sigma[\tilde{r}_m]} = p(\tilde{C}F_j)(1 + r_f)$$

$$\frac{E[\tilde{C}F_j] - (E[\tilde{r}_m] - r_f)\frac{\sigma[\tilde{C}F_j]}{\sigma[\tilde{r}_m]}}{1 + r_f} = p(\tilde{C}F_j) \tag{2}$$